M.U. M.Sc. '94

Q.No → Use the method of Contour integration to evaluate:

$$\int_0^\infty \frac{\log(1+x^2)}{1+x^2}\,dx.$$

or, Prove by Contour integration, that

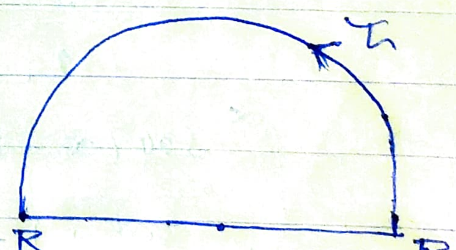
$$\int_0^\infty \frac{\log(1+x^2)}{1+x^2}\,dx = \pi \log 2.$$

Proof:- Let us Consider the integral

$$\int_C \frac{\log(z^2+i)}{z^2+1}\,dz = \int_C f(z)\,dz$$

Where C is Contour Consisting of the real axis from $-R$ to R and the semi-circle Γ of large radius R in the upper half of the z-plane.

$$\therefore \int_C f(z)\,dz = \int_{-R}^{R} f(x)\,dx + \int_\Gamma f(z)\,dz$$



$$= 2\pi i \sum R^+$$

Where $\sum R^+$ is the Some of the residue of $f(z)$ at Poles of $f(z)$ within C, we have

$$\lim_{z\to\infty} z f(z) = \lim_{z\to\infty} \frac{z\cdot\log(z^2+i)}{z^2+1} = 0$$

$$\therefore \lim_{R\to\infty} \int_\Gamma f(z)\,dz = 0 \therefore \int_C f(z)\,dz = \int_{-\infty}^{\infty} f(x)\,dx = 2\pi i \sum R^+$$

The Poles of $f(z)$ are given by $z^2 = -1 = i^2$ of which only lies within C, the residue of $f(z)$ at $z = i$ will be given by

$$\lim_{z \to i} (z-i) f(z) = \lim_{z \to i} \frac{(z-i)\log(z+i)}{z^2+1}$$

$$= \lim_{z \to i} \frac{\log(z+i)}{z+i}$$

$$= \frac{\log 2i}{2i} = \frac{\log 2 + \log i}{2i}$$

$$= \frac{\log 2 + \log e^{i\pi/2}}{2i} = \frac{\log 2 + i\pi/2}{2i}$$

$$\therefore \int_C f(z)\,dz = \int_{-\infty}^{\infty} f(x)\,dx = 2\pi i \sum R^+$$

$$= \frac{2\pi i \left(\log 2 + \frac{i\pi}{2}\right)}{2i} = \pi\left(\log + \frac{i\pi}{2}\right)$$

$$\therefore \int_{-\infty}^{\infty} \frac{\log(x+i)}{x^2+1}\,dx = \pi\left(\log + \frac{i\pi}{2}\right)$$

$$\therefore \int_{-\infty}^{\infty} \frac{\frac{1}{2}\log(x^2+1) + i\tan^{-1}\frac{1}{x}}{x^2+1}\,dx$$

$$= \pi \log 2 + \frac{i\pi^2}{2}$$

Equating the real Part from both Sides, we have

$$\frac{1}{2}\int_{-\infty}^{\infty}\frac{\log(x^2+1)}{x^2+1}\,dx = \pi\log 2$$

$$\therefore \frac{1}{\cancel{2}}\cdot\cancel{2}\int_{0}^{\infty}\frac{\log(x^2+1)}{x^2+1}\,dx = \pi\log 2.$$

i.e. $$\int_{0}^{\infty}\frac{\log(x^2+1)}{x^2+1}\,dx = \pi\log 2.$$

M.Sc. 94

Q No → Use the method of Contour integration to evaluate of the following:

(i) $\displaystyle\int_0^\infty \frac{\sin \alpha x}{x}\,dx, \ \alpha > 0$

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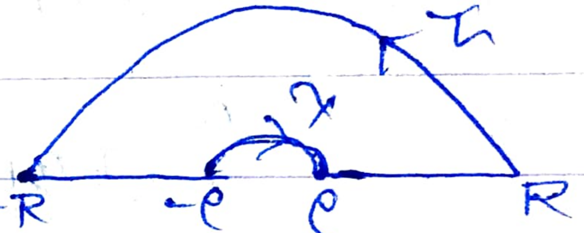
(ii) $\displaystyle\int_0^\infty \frac{x \sin x}{x^2 + a^2}\,dx, \ a > 0$

(i) We Consider the

$$\int_C \frac{e^{iaz}}{z} dz = \int_C f(z)\, dz$$



Where C is Contour Consisting of the real axis from $-R$ to $-\rho$ the Semi-Circle γ of Small radius ρ the real axis from ρ to R

and the Semi-Circle Γ of large radius R in the upper half of Z-Plane.

$$\therefore \int_C f(z)\,dz = \int_{-R}^{-\rho} f(x)\,dx + \int_{\gamma} f(z)\,dz + \int_{\rho}^{R} f(x)\,dx + \int_{\Gamma} f(z)\,dz = 0$$

$\because f(z)$ has no Pole Within C Since,

$\lim_{z\to\infty} \frac{1}{z} = 0$ by Jordan's lemma, We have

$$\lim_{R\to\infty} \int_{\Gamma} \frac{e^{i\alpha z}}{z}\,dz = \lim_{R\to\infty} \int_{\Gamma} f(z)\,dz = 0$$

$$\text{Again, } \lim_{z\to 0} z f(z) = \lim_{z\to 0} z \cdot \frac{e^{i\alpha z}}{z}$$

$$= \lim_{z\to 0} e^{i\alpha z} = e^0 = 1.$$

$$\therefore \lim_{\rho\to 0} \int_{\gamma} f(z)\,dz = i\pi(0-1) = -i\pi$$

$$\therefore \int_{-\infty}^{0} f(x)\,dx - i\pi + \int_{0}^{\infty} f(x)\,dx = 0$$

$$\therefore \int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{\infty} \frac{e^{i\alpha x}}{x}\,dx = i\pi$$

$$\therefore \int_{-\infty}^{\infty} \frac{\cos\alpha x + i\sin\alpha x}{x}\,dx = i\pi$$

$\therefore$ Equating the imaginary Part of both Sides, We have

$$\therefore \int_{-\infty}^{\infty} \frac{\sin\alpha x}{x}\,dx = \pi \quad \therefore 2\int_{0}^{\infty} \frac{\sin\alpha x}{x}\,dx = \pi$$

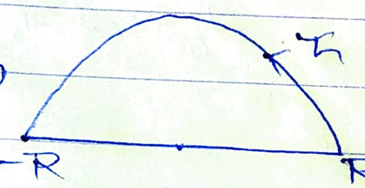
$$\therefore \int_{0}^{\infty} \frac{\sin\alpha x}{x}\,dx = \frac{\pi}{2}$$

(ii) We consider the

$$\int_C \frac{z e^{iz}}{z^2+a^2} = \int_C f(z)\,dz$$

Where C is contour consisting of the real axis from $-R$ to R and the semi-circle Γ of large radius R in the upper half of the z-plane.

$$\therefore \int_C f(z)\,dz = \int_{-R}^{R} f(x)\,dx + \int_\Gamma f(z)\,dz = 0$$



$$\therefore \int_C f(z)\,dz = \int_{-\infty}^{\infty} f(x)\,dx = 2\pi i \sum R^+$$

Where $\sum R^+$ is the sum of the ~~resid~~ residues at poles of $f(z)$ within C. The poles of $f(z)$ are given by $z^2+a^2=0$ i.e. $z=\pm ai$ of which only $z=ai$ lies within C. The residue of $f(z)$ at $z=ai$ will be given by

$$\underset{z\to ai}{Lt}\,(z-ai)f(z) = \underset{z\to ai}{Lt}\,\frac{(z-ai)ze^{iz}}{z^2+a^2}$$

$$= \underset{z\to ai}{Lt}\,\frac{ze^{iz}}{z+ai} = \frac{ze^{iz}}{2ai} = \frac{e^{-a}}{2}$$

$$\therefore \int_C f(z)\,dz = \int_{-\infty}^{\infty} f(x)\,dx = 2\pi i \sum R^+$$

$$= 2\pi i \cdot \frac{e^{-a}}{2} = \pi i e^{-a}$$

Equating real parts from both sides, we have

$$\int_{-\infty}^{\infty} \frac{x \sin x}{x^2 + a^2}\, dx = \pi e^{-a}$$

$$\therefore 2\int_{0}^{\infty} \frac{x \sin x}{x^2 + a^2} = \pi e^{-a}$$

$$\therefore \int_{0}^{\infty} \frac{x \sin x}{x^2 + a^2} = \frac{\pi e^{-a}}{2}.$$